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Generating functions and elementary Young tableaux

M Couture†, C J Cummins‡ and R T Sharp§

† Theoretical Physics Branch, Chalk River Nuclear Laboratory, Chalk River, Ontario, Canada
‡ Department of Mathematics, Concordia University, Montreal, Quebec, Canada
§ Department of Physics, McGill University, Montreal, Quebec, Canada

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Abstract. When constructing polynomial bases for group representations, one is led naturally to consider a 'stretched product' of states. Elementary multiplets are states that cannot be expressed as the stretched product of any other two states, and they can be identified with terms occurring in certain generating functions. In this paper we define an analogous stretched product of Young tableaux together with elementary tableaux. These may be used in a manner similar to elementary multiplets to construct generating functions. We consider several examples where this method can be applied and in particular we give new large-N (rank-free) branching rule generating functions for the group-subgroup pairs $SU(n)\times SU(m)\subset SU(n+m)$ (first four labels non-zero), $SO(n)\subset SU(n)$ (first three labels non-zero) and $Sp(2n)\subset SU(2n)$ (first five labels non-zero). Our method of finding these generating functions uses the theory of Grobner bases which allows us to make precise the notion of compatibility relations or forbidden products of elementary tableaux.

1. Introduction

Generating functions are a convenient and useful way of both presenting and calculating group theoretical information. Examples are provided by branching rule generating functions (Patera and Sharp 1979, 1982, Saint-Aubin 1980, Gaskell and Sharp 1981), weight generating functions (Gaskell *et al* 1978), orbit generating functions (Michel *et al* 1988) and generating functions for plethysms (Patera and Sharp 1980). In each case the generating function is a rational function of a set of auxiliary variables; the branching rule multiplicities, weight multiplicities etc, then appear as the coefficients in its power series expansion. Generating functions provides 'global' information in the sense that they contain information about all irreducible representations (or in some cases infinite subsets of all irreducible representations).

Another technique which is also useful, particularly in the case of the classical groups, is that of Young tableaux (Littlewood 1950, Stanley 1980, King and El-Sharkaway 1983, Baclawski 1983, King 1975, 1976, Black and Wybourne 1983, Yang and Wybourne 1986, Cummins 1987, Tokuyama 1986). The methods have also been applied to the exceptional groups (King and Al-Qubanchi 1978, 1981a, b, Wybourne 1973, 1979, 1984, Wybourne and Bowick 1977) and supergroups (Dondi and Jarvis 1981, Balantekin and Bars 1981a, b, 1982, Bars *et al* 1983, Farmer and Jarvis 1984, Bars 1984, Delduc and Gourdin 1984, 1985, Morel *et al* 1985, Gourdin 1986, Cummins and King 1987a, b). Here the emphasis is different. Combinatorial procedures are given that involve filling the boxes of some Young diagram with integers subject to certain constraints. The number of ways of doing this is then the required multiplicity (possibly after some modifications, see King (1975)). The analysis in this case is 'local' when compared with the generating function approach since one calculates multiplicities for particular representations.

Techniques have been developed to find generating functions that involve polynomial bases for group representations (Moshinsky and Devi 1969, Sharp and Lam 1969). In this approach it is natural to consider stretched products of states and so to define elementary multiplets as states that are not the product of any other two states. From these elementary multiplets and their syzygies (algebraic relations) it is possible to deduce the required generating functions. We refer the reader to the references above for more information.

In this paper we shall describe how the method of Young diagrams may be extended to calculate generate functions. In practice this method is simple to apply, mainly because it picks out a unique set of elementary tableaux (which play the same role as elementary multiplets) and a corresponding set of syzygies, which may be determined 'locally'. Computer programs may be written to perform these calculations. The key idea is to define a stretched product of Young tableaux which can be shown in many cases to preserve the constraints mentioned above. Elementary tableaux are then those that cannot be expressed as the product of two distinct tableaux. They may be found using algorithms developed for the solution of linear Diophantine equations (Stanley 1973, Huet 1978). Bases for the syzygies (algebraic relations) between these elementary tableaux may then be found using the theory of Grobner bases (Buchberger 1989) and from these we may find a generating function. The use of Grobner bases in this context makes rigorous the ideas of compatibility relations or forbidden products of elementary tableaux and their use in calculating generating functions (cf Sharp and Lam 1969). Essentially the idea is that any tableau may be constructed as a product of elementary tableaux, but in general this expression is not unique. Uniqueness may be restored by forbidding the product of some of the elementary multiplets. From this unique decomposition it is a straightforward task to find a generating function. The problem is to know when all forbidden products have been found and if a given set of forbidden products gives rise to a unique decomposition of every tableau (these two questions are not identical). These problems are solved by theorems in the theory of Grobner bases which we quote in section 4.

For the method to be applicable it is necessary that a Young tableau technique exists for computing the multiplicity of interest which does not involve modification rules (some $SO(2n)$ algorithms also do not work). Although this is a limitation, particularly as regards branching rules, new Young tableau techniques that do not require modification rules (Tokuyama 1986) seem well suited to the method. We are also limited by the complexity of the final generating functions; the larger the generating function the more difficult it is to calculate. This problem, however, is one that applies to any generating function and not just to our particular method of calculating them.

The first use of Young tableaux for finding elementary multiplets appears in the work of Moshinsky and Devi (1969), where the branching $SO(3) \subset SU(3)$ was considered. Here elementary tableaux were called elementary permissible diagrams. The character generators of Stanley (1980), Baclawski (1983) and King and El-Sharkaway (1983, 1984), are also closely related to the problems we shall be considering, and we shall have more to say about them in section 5.

In section 2 the notation and some basic results of Young tableau theory will be reviewed. In section 3 we describe the stretched product of tableaux and show that in the cases we consider it is well defined. Section 4 contains some results on rank-free generating functions, together with an explanation of the properties of Grobner bases that we have used and section 5 contains examples of other types of generating function that can be found using the same techniques. The final section contains some comments.

2. Young diagrams and Young tableaux

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a weakly decreasing sequence of positive integers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k > 0$. Associated with each partition λ is a Young diagram F^λ. This is an array of left-justified boxes in the plane such that the number of boxes in the ith row is λ_i. The depth $l(\lambda)$ of the partition λ is defined to be the number of parts of λ, for example $l(2, 1, 1) = 3$. Thus $l(\lambda)$ is the number of rows of F^λ and we set $l(F^\lambda) = l(\lambda)$.

We say that the Young diagram F^λ contains the Young diagram F^σ if $l(\lambda) \geqslant l(\sigma)$ and $\lambda_i \geqslant \sigma_i$, $1 \leqslant i \leqslant l(\sigma)$; in other words F^σ may be placed entirely inside F^λ in the top left hand corner as illustrated in figure 1. If F^λ contains F^σ then we write $F^\sigma \subset F^\lambda$. Finally we define $|\lambda| = \Sigma_i \lambda_i$ to be the size of the partition λ.

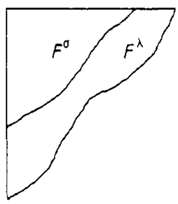

Figure 1. F^λ contains F^σ.

It is well known that the irreducible representations of the classical groups U(n), SU(n), SO(n) and Sp($2n$) can be labelled by partitions. This labelling and its relationship with Dynkin's labels may be found in table 1; for more details see King (1975). In fact we shall not need these results in their full generality. We simply note that if only the first k Dynkin labels of a representation are non-vanishing and if k is not too large compared with the rank of the corresponding group, then the relationship between the Dynkin labels a_j and partition labels λ_i simplifies to

$$\lambda_i = \sum_{j=i}^{k} a_j. \tag{2.1}$$

This is equivalent to the statement that a_j is the number of columns of length j in F^λ.

Recall that roughly speaking a Young tableau T^λ is obtained by filling the boxes of F^λ with integers subject to some constraints. In this sense the computation of weight multiplicities, branching multiplicities, multiplicities of tensors in enveloping algebras etc amounts to the enumeration of certain kinds of Young tableaux. In the remainder of this section we shall state some of these enumeration theorems together with the relevant definitions. The reader should note that in the literature there is a large amount of variation in notation.

Definition 2.1. A Young taleau T^λ is a Young diagram F^λ with some or all its boxes filled with integers (excluding zero) in such a way that the unfilled boxes form a Young diagram F^σ. Clearly we must have $F^\sigma \subset F^\lambda$. We shall call the partition σ the content

Table 1. Partition labels and corresponding Dynkin labels for several groups.

Group	Partition label	Dynkin label		
U(n), SU(n)	$\{\bar{\mu}; \lambda\}$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant p-1$	$\lambda_p = -(s - t - qa_p)/n$
	$l(\mu) + l(\lambda) \leqslant n$	$a_{n-i} = \mu_i - \mu_{i+1}$	$1 \leqslant i \leqslant q-1$	$\mu_q = (s - t + pa_p)/n$
		$a_p = \lambda_p + \mu_q$		$s = \Sigma_{i=1}^{p-1} ia_i - \Sigma_{i=1}^{q-1} ia_{n-i}$
		$p = l(\lambda) \qquad q = n - p$		$\lambda_i = \Sigma_{k=i}^{p-1} a_i + \lambda_p$
		$t = \Sigma_{i=1}^{p} \lambda_i - \Sigma_{i=1}^{q} \mu_i$		$\mu_i = \Sigma_{k=i}^{q-1} a_{n-k} + \mu_q$
		t is the U(1) weight		
SU(n)	$\{\lambda\}$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-2$	$\lambda_i = \Sigma_{k=i}^{n-1} a_k$
	$l(\lambda) \leqslant n-1$	$a_{n-1} = \lambda_{n-1}$		
Sp($2n$)	$\langle\lambda\rangle$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-1$	$\lambda_i = \Sigma_{k=i}^{n} a_k$
	$l(\lambda) \leqslant n$	$a_n = \lambda_n$		
SO($2n+1$)	$[\lambda]$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-1$	$\lambda_i = \Sigma_{k=i}^{n-1} a_k + \frac{1}{2}a_n$
	$l(\lambda) \leqslant n$	$a_n = 2\lambda_n$		a_n even
	$[\Delta; \lambda]$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-1$	$\lambda_i = \Sigma_{k=i}^{n-1} a_k + \frac{1}{2}(a_n - 1)$
	$l(\lambda) \leqslant n$	$a_n = 2\lambda_n + 1$		a_n odd
SO($2n$)	$[\lambda]$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-2$	$\lambda_i = \Sigma_{k=i}^{n-1} a_k$
	$l(\lambda) < n$	$a_{n-1} = a_n = \lambda_{n-1}$		$\lambda_n = 0 \qquad a_n = a_{n-1}$
	$[\lambda]_+$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-2$	$\lambda_i = \Sigma_{k=i}^{n-2} a_k + \frac{1}{2}(a_{n-1} + a_n)$
	$l(\lambda) = n$	$a_{n-1} = \lambda_{n-1} - \lambda_n$		$\lambda_{n-1} = \frac{1}{2}(a_{n-1} + a_n)$
		$a_n = \lambda_{n-1} + \lambda_n$		$\lambda_n = \frac{1}{2}(-a_{n-1} + a_n)$
				$a_n > a_{n-1}$, $a_{n-1} + a_n$ even
	$[\lambda]_-$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-2$	$\lambda_i = \Sigma_{k=i}^{n-2} a_k + \frac{1}{2}(a_{n-1} + a_n)$
	$l(\lambda) = n$	$a_{n-1} = \lambda_{n-1} + \lambda_n$		$\lambda_{n-1} = \frac{1}{2}(a_{n-1} + a_n)$
		$a_n = \lambda_{n-1} - \lambda_n$		$\lambda_n = \frac{1}{2}(a_{n-1} - a_n)$
				$a_n < a_{n-1}$, $a_{n-1} + a_n$ even
	$[\Delta; \lambda]_+$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-2$	$\lambda_i = \Sigma_{k=i}^{n-2} a_k + \frac{1}{2}(a_{n-1} + a_n - 1)$
	$l(\lambda) \leqslant n$	$a_{n-1} = \lambda_{n-1} - \lambda_n$		$\lambda_{n-1} = \frac{1}{2}(a_{n-1} + a_n - 1)$
		$a_n = \lambda_{n-1} + \lambda_n + 1$		$\lambda_n = \frac{1}{2}(-a_{n-1} + a_n - 1)$
				$a_n > a_{n-1}$, $a_{n-1} + a_n$ odd
	$[\Delta; \lambda]_-$	$a_i = \lambda_i - \lambda_{i+1}$	$1 \leqslant i \leqslant n-2$	$\lambda_i = \Sigma_{k=i}^{n-2} a_k + \frac{1}{2}(a_{n-1} + a_n - 1)$
	$l(\lambda) \leqslant n$	$a_{n-1} = \lambda_{n-1} + \lambda_n + 1$		$\lambda_{n-1} = \frac{1}{2}(a_{n-1} + a_n - 1)$
		$a_n = \lambda_{n-1} - \lambda_n$		$\lambda_n = \frac{1}{2}(a_{n-1} - a_n - 1)$
				$a_n < a_{n-1}$, $a_{n-1} + a_n$ odd

of the tableau T^λ and we say that T^λ has shape $\lambda - \sigma$. The sequence $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_k)$, where ϑ_i = number of occurrences of i minus number of occurrences of $-i$ in T^λ, is called the weight of T^λ. The depth $l(T^\lambda)$ of T^λ is the number of rows in the corresponding Young diagram. We also set

$$\boldsymbol{T} = \{T \mid T \text{ is a Young tableau}\}$$

$$\boldsymbol{T}^+ = \{T \in \boldsymbol{T} \mid T \text{ has only positive entries}\}$$

$$\boldsymbol{T}^k = \{T \in \boldsymbol{T} \mid l(T) \leqslant k, \text{ and } T \text{ has only entries } \pm 1, \pm 2, \ldots, \pm k\}$$
$$\boldsymbol{T}^{k+} = \boldsymbol{T}^k \cap \boldsymbol{T}^+.$$

When imposing conditions on tableaux it is convenient to fix a total ordering of the integers occurring in these tableaux. Unless otherwise stated, we shall use the ordering

$$-1 < 1 < -2 < 2 < \ldots < -k < k < \ldots \tag{2.2}$$

which reduces to the usual ordering on $\boldsymbol{T}^+$. An exception is the case of Tokuyama's sympletic tableaux (see below).

Definition 2.2. A tableau $T \in \boldsymbol{T}$ is said to be weakly normal if the entries in each row of T are weakly increasing (in the relevant ordering!) from left to right. We denote by $\boldsymbol{WN}$ the set of all weakly normal tableaux.

Definition 2.3. A tableau $T \in \boldsymbol{T}$ is said to be standard if it is weakly normal and the entries in each column are strictly increasing from top to bottom. We denote by $\boldsymbol{S}$ the set of all standard tableaux.

King (1976) and King and El-Sharkaway (1983) have defined Young tableaux for computing the weight multiplicities of the representations of each of the classical groups. The treatment of the various orthogonal groups is, however, quite involved and we shall give only the definition of unitary and symplectic tableaux.

Definition 2.4. We call the elements of $\boldsymbol{S}^{k+}$ k-unitary tableaux or simply unitary tableaux and denote this set by $\boldsymbol{U}^k$.

Definition 2.5. A tableau T is said to be a k-symplectic tableau or simply a sympletic tableau if (i) $T \in \boldsymbol{S}^k$ and (ii) $r(i) \leqslant i$ and $r(-i) \leqslant i$, $1 \leqslant i \leqslant k$, where $r(i)$ and $r(-i)$ are the lowest rows containing i and $-i$ respectively.

We denote by $\boldsymbol{Sp}^k$ the set of all k-sympletic tabeaux.

Definition 2.6. Let $T \in \boldsymbol{T}$; then we call the sequence of numbers obtained by reading the entries of T starting from the top right and reading from right to left and from top to bottom the word of T, which is denoted by $W(T)$. We denote by $c_T(i, p)$ the number of occurrences of the integer i in the first p terms of $W(T)$ and by $|W(T)|$ the number of terms in $W(T)$.

Definition 2.7. Let $T \in \boldsymbol{T}^+$ and let m be the largest integer occurring in $W(T)$; then T is said to satisfy the LR (Littlewood-Richardson) condition if

$$c_T(i, p) \geqslant c_T(i+1, p) \qquad 1 \leqslant i \leqslant m-1 \qquad 1 \leqslant p \leqslant |W(T)|$$

and we call T an LR tableau if it is also standard. We denote by $\boldsymbol{LR}$ the set of all LR tableaux.

Comment. The LR condition implies that the weight of T is a partition, and also that if $i > 1$ then i does not occur in rows 1 to $i-1$.

Recently Tokuyama (1986) has introduced a new type of tableau that may be used to compute branching multiplicities of $\mathrm{Sp}(2n) \subset \mathrm{SU}(2n)$. They are defined as follows.

Definition 2.8. Let $T \in T^{2k}$; then we say that T is a k-Tokuyama tableau if it satisfies the following conditions:

(i) the content of T vanishes (every box is filled);

(ii) only the integers $\pm 1, \pm 2, \ldots, \pm k$ occur in T;

(iii) T is standard under the ordering $1 < 2 < \ldots < k < -k < -(k-1) < \ldots < -2 < -1$;

(iv) the positive entries satisfy the LR condition $c_T(i, p) \geqslant c_T(i+1, p)$, $1 \leqslant i \leqslant k-1$, $1 \leqslant p \leqslant |W(T)|$;

(v) if $d_T(i, p) = c_T(i, p) - c_T(-i, p)$ then $d_T(i, p) \geqslant d_T(i+1, p) \geqslant 0$, $1 \leqslant i \leqslant k-1$, $1 \leqslant p \leqslant |W(T)|$.

Note. Condition (iv) may be replaced by the condition that the positive entries of T form a canonical tableau. This is to say that the entry i occurs only in the ith row and the boxes corresponding to these entries form a Young diagram.

These various tableaux may be used to compute many types of multiplicities.

Theorem 2.1 (King and El-Sharkaway 1983). (i) The multiplicity of the weight $\tilde{\vartheta} = (\tilde{\vartheta}_1, \tilde{\vartheta}_2, \ldots, \tilde{\vartheta}_{k-1})$ in the representation of $\mathrm{SU}(k)$ labelled by the partition λ, $l(\lambda) \leqslant k-1$, is given by the number of k-unitary tableaux T^λ of shape λ and such that the weight $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_k)$ of T^k satisfies $\tilde{\vartheta}_1 = \vartheta_1 - \vartheta_2$, $\tilde{\vartheta}_2 = \vartheta_2 - \vartheta_3, \ldots, \tilde{\vartheta}_{k-1} = \vartheta_{k-1} - \vartheta_k$.

(ii) The multiplicity of the weight $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_k)$ in the representation of $\mathrm{Sp}(2k)$ labelled by the partition λ, $l(\lambda) \leqslant k$ is given by the number of k-symplectic tableaux of shape λ.

Theorem 2.2 (Littlewood 1950). (i) Consider the restriction of the representation of $\mathrm{SU}(n+m)$ labelled by the partition λ to the canonical subgroup $\mathrm{SU}(n) \times \mathrm{SU}(m)$. If $l(\lambda) \leqslant \min(n-1, m-1)$ then the multiplicity of the representation of $\mathrm{SU}(n) \times \mathrm{SU}(m)$ labelled by the pair of partitions (σ, τ) is the number of LR tableaux of shape $\lambda - \sigma$ and weight τ.

(ii) Consider the restriction of the representation of $\mathrm{SU}(n)$ labelled by the partition λ to the canonical subgroup $\mathrm{SO}(n)$. If $l(\lambda) \leqslant n/2 - 1$ for even n and $l(\lambda) \leqslant \frac{1}{2}(n-1)$ for odd n, then the multiplicity of the representation of $\mathrm{SO}(n)$ labelled by the partition σ is the number of LR tableaux of shape $\lambda - \sigma$ and such that the weight has only even parts.

(iii) Consider the restriction of the representation of $\mathrm{SU}(n)$ labelled by the partition λ to the canonical subgroup $\mathrm{Sp}(2n)$. If $l(\lambda) \leqslant n$, then the multiplicity of the representation of $\mathrm{Sp}(2n)$ labelled by the partition σ is the number of LR tableaux of shape $\lambda - \sigma$ and such that the weight (or more precisely the corresponding Young diagram) has only even columns.

Comment. We call the branching rules described in the previous theorem rank-free branching rules since they apply to any rank provided the constraints referred to in the theorems are satisfied. These restrictions may be removed if we take modification rules into account. For the case of the branching to $\mathrm{SU}(n) \times \mathrm{SU}(m)$ the modification rule is very simple, we simply delete all representations (σ), (τ) where either $l(\sigma) > n$ or $l(\tau) > m$. If the equality holds we remove all columns of length n or m respectively. The modification rules for SO(n) and Sp($2n$) are a little more complicated and details may be found in King (1975). For the case of Sp($2n$), Tokuyama (1986) has devised a different type of tableau that requires no modification rules.

Let U(g) be the universal enveloping algebra of the Lie algebra g of the Lie group G. U(g) is G-module and is in fact isomorphic as a G-module to S(g) the symmetric algebra of g considered as the adjoint representation of G. We may thus decompose U(g) into a direct sum of irreducible G-modules which respects the grading of U(g) by degree. To use Young tableaux to compute the multiplicities of these irreducible constituents of the universal enveloping algebra of SU(n) we first make the following definition.

Definition 2.9. Let μ be a partition and n a positive integer such that $n \geq l(\mu)$, then the n-complement of μ, τ say, is a partition such that

$$\tau_i' = n - \mu'_{\mu_1 - i + 1} \qquad 1 \leq i \leq \mu_1.$$

The following theorem is easier to state for U(k) than SU(k).

Theorem 2.3. The multiplicity of the representation of U(k) labelled by the partition λ, $l(\lambda) \leq k$, of degree n in U(U(k)) is given by the number of LR tableaux of shape $\lambda - \mu$ and weight τ where μ is any partition of n, and τ is the k-complement of μ.

Note. To obtain the multiplicities in SU(k) we must count the number of LR tableaux of shape $\lambda^* - \mu$ where λ^* is any partition obtained from λ by adding columns of length k to the corresponding Young diagram. We must also, however, exclude all tableaux whose first column is of the form

$$\begin{array}{|c|} \hline \\ \hline 1 \\ \hline 2 \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|} \hline k-2 \\ \hline k-1 \\ \hline \end{array}\,. \tag{2.3}$$

This corresponds to eliminating the degree-1 scalar which arises from the branching of the adjoint of U(k) to SU(k). For the purposes of finding syzygies and generating functions as described in sections 4 and 5, the easiest procedure is to retain all tableaux, and then simply delete the factor $(1-U)^{-1}$ from the generating function. The above tableau (2.3) must then be removed from the list of elementary tableaux.

Tokuyama (1986) has described a remarkable method of finding the branching multiplicities for the branching Sp($2k$) $\subset$ SU($2k$), which requires no modification rules. The result is contained in the following theorem.

Theorem 2.4. Consider the restriction of the representation of SU($2k$) labelled by the partition λ, $l(\lambda) \leq 2k-1$, to the canonical Sp($2k$) subgroup. The multiplicity of the

Sp($2k$) representation labelled by the partition $\sigma, l(\sigma) \leqslant k$, is the number of k-Tokuyama tableaux of shape λ and weight σ.

Comment. Theorem 4 also applies in the case $l(\lambda) = 2k$, in this case, however, the corresponding representation of SU($2k$) is equivalent to a representation with $l(\lambda) < 2k$.

3. Stretched products of Young tableaux

In order to introduce the stretched product of tableaux we first make the following definition.

Definition 3.1. The stretched product of two partitions, $\lambda.\mu$, is a partition with parts $(\lambda.\mu)_i = \lambda_i + \mu_i$ (if i exceeds the length of either partition then the corresponding part λ_i or μ_i is taken to be zero). We extend this definition to the stretched product of Young diagrams by setting $F^\lambda.F^\mu = F^{\lambda.\mu}$.

Note. This stretched product is usually called the sum of the partitions λ and μ, and is written $\lambda + \mu$. We have chosen the product notation since this is more in harmony with the standard notation of elementary multiplets.

Definition 3.2. Let T^λ and T^μ be weakly normal Young tableaux and let $n_\lambda(r, s)$ (respectively $n_\mu(r, s)$) be number of entries of the integer r occurring the sth row of T^λ (respectively T^μ); for convenience, the unfilled boxes are considered to be filled with zeros. The stretched product $T^\lambda.T^\mu$ is defined to be the weakly normal Young tableau obtained by placing integers in the boxes of $F^{\lambda.\mu}$ in such a way that

$$n(r, s) = n_\lambda(r, s) + n_\mu(r, s)$$

where $n(r, s)$ is the number of entries r in the sth row of $T^\lambda.T^\mu$.

Example.

$$\begin{array}{|c|c|c|}\hline \ & 1 & 1 \\ \hline 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \cdot \begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|c|c|}\hline \ & \ & \ & 1 & 1 & 1 \\ \hline 1 & 2 & 2 & \multicolumn{3}{c}{} \\ \cline{1-3} 2 & 3 & \multicolumn{4}{c}{} \\ \cline{1-2} 3 & \multicolumn{5}{c}{} \\ \cline{1-1} \end{array}\ .$$

Comment. The stretched product makes $\boldsymbol{WN}$ into a commutative semigroup and $\boldsymbol{WN}^+$, $\boldsymbol{WN}^k$ and $\boldsymbol{WN}^{k+}$ are subsemigroups of $\boldsymbol{WN}$.

Definition 3.3. Let X be a subsemigroup of $\boldsymbol{WN}$; then an element $a \in X$ is said to be an elementary tableau (of X) if a is not the stretched product of two elements of X.

Comment. Clearly the elementary tableaux generate X, but in general there need not be a finite number of elementary tableaux and the expression for an arbitrary element of X need not have a unique expression as a product of elementary tableaux.

In the rest of this section we will establish that the subsets of $\boldsymbol{WN}$ introduced in the last section are in fact closed under taking stretched products. In some simple cases it is also possible to state the general form of the elementary tableaux (or at least an algorithm for their construction). In general, however, the task of classifying all elementary tableaux seems to be difficult and we shall be content to describe some particular cases in the next two sections.

Proposition 3.1. Let T_1, $T_2 \in \boldsymbol{S}^k$; then the product $T_1 . T_2 \in \boldsymbol{S}^k$.

Proof. Clearly T_1, $T_2 \in \boldsymbol{WN}^k$ by construction; assume that it is not standard. Then for some box with entry η in the ith row of $T_1 . T_2$ the box above in the $(i-1)$th row must contain an equal or larger symbol. Thus the number of boxes containing a symbol η or smaller plus the number of unfilled boxes in row i is strictly greater than the number of boxes containing a symbol smaller than η plus the number of unfilled boxes in row $i-1$. On the other hand, for each of T_1 and T_2 this inequality is reversed, or equality holds, since both are standard. But adding these two inequalities gives us a contradiction and so $T_1 . T_2$ is standard.

Corollary 3.1. If T_1, $T_2 \in \boldsymbol{U}^k$ (respectively $\boldsymbol{Sp}^k$) then $T_1 . T_2 \in \boldsymbol{U}^k$ (respectively $\boldsymbol{Sp}^k$).

Proof. $\boldsymbol{U}^k = \boldsymbol{S}^{k+}$ and the condition of having only positive entries is clearly preserved. Similarly $\boldsymbol{Sp}^k \subset \boldsymbol{S}^k$ and the additional constraints are again clearly preserved by the stretched product.

Proposition 3.2. Let T_1, $T_2 \in \boldsymbol{WN}^+$ satisfy the LR condition, then $T_1 . T_2$ satisfies the LR condition.

Proof. Let us assume that $T_1 . T_2$ does not satisfy the LR condition, then there exist i and p such that $C_{T_1 . T_2}(i, p) < C_{T_1 . T_2}(i+1, p)$. On the other hand, the corresponding subword arises from the combination of two subwords from T_1 and T_2 in each of which the opposite inequality or equality holds. Clearly the number of occurrences of the symbol i is just the sum of the occurrences in each subword and so we have a contradiction.

Corollary 3.2. If T_1, $T_2 \in \boldsymbol{LR}$ then $T_1 . T_2 \in \boldsymbol{LR}$.

Proof. By propositions 3.1 and 3.2 the required properties of standardness and of satisfying the LR conditions are both preserved by the stretched product.

Corollary 3.3. If T_1 and T_2 are two LR tableaux with weights which have only even rows (respectively columns) then $T_1 . T_2$ is an LR tableau whose weight has only even rows (respectively columns).

Proof. The weight of $T_1 . T_2$ is the stretched product of the weights of T_1 and T_2. It is clear that this product of Young diagrams preserves the required properties of rows and columns.

Definition 3.4. It will be convenient to denote the set of all LR tableaux with weights having only even rows (respectively columns) by $\boldsymbol{LR}_{\mathrm{o}}$ (respectively $\boldsymbol{LR}_{\mathrm{sp}}$), and similarly $\boldsymbol{LR}^k_{\mathrm{o}}$ (respectively $\boldsymbol{LR}^k_{\mathrm{sp}}$) are the LR tableaux of depth less than or equal to k whose weights have only even rows (respectively columns).

We now show that for each fixed m, $\boldsymbol{LR}^m$, $\boldsymbol{LR}^m_{\mathrm{o}}$ and $\boldsymbol{LR}^m_{\mathrm{sp}}$ have only a finite number of elementary tableaux. First we recall a result given in Stanley (1983) concerning the solutions in non-negative integers of linear systems of equations over the integers. Let ϕ be an $r \times n$ matrix with integer entries, $r \leqslant n$ and $\mathrm{rank}(\phi) = r$. Define

$$E_\phi = \{\beta \in \mathbb{N}^n | \phi\beta = 0\}$$

then E_ϕ is a monoid and also has the following property.

Theorem 3.1 (Stanley (1983), theorem 3.1). E_ϕ is a finitely generated monoid.

The task is thus to find a parametrisation by non-negative integers subject to linear constraints of the set of Young tableaux of interest. For example, let us adopt the parametrisation shown in figure 2. In figure 2, α_1, $n(1,1)$, α_2, $n(1,2), \ldots$ are arbitrary non-negative integers (note that this means we do not necessarily have a Young tableau since the row lengths may not be weakly decreasing). We can then make the following proposition.

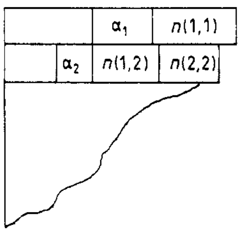

Figure 2. The parametrisation adopted for the purposes of proposition 3.3.

Proposition 3.3. The linear constraints for column strictness (including the constraint that the row lengths be weakly decreasing) are

$$\sum_{i=1}^{k-1} n(i, j-1) - \sum_{i=1}^{k} n(i,j) + \alpha_{j-1} - s_{k,j} = 0 \qquad 1 \leqslant k \leqslant j \leqslant l(T^\lambda) \quad j \neq 1 \quad s_{k,j} \geqslant 0 \tag{3.1}$$

and the conditions for satisfying the Littlewood–Richardson condition are

$$\sum_{i=k}^{j} n(k-1, i-1) - \sum_{i=k}^{j} n(k, i) - t_{k,j} = 0 \qquad 2 \leqslant k \leqslant j \leqslant l(T^\lambda) \quad t_{k,j} \geqslant 0. \tag{3.2}$$

As an example, consider the case of two rows,

	α_1	$n(1,1)$
α_2	$n(1,2)$	$n(2,2)$

.

The constraint equations become

$$\begin{aligned} &-n(1,2) + \alpha_1 - s_{1,2} = 0 \\ &n(1,1) - n(1,2) - n(2,2) + \alpha_1 - s_{2,2} = 0 \\ &n(1,1) - n(2,2) - t_{2,2} = 0 \end{aligned} \tag{3.3}$$

corresponding to the inequalities:

$\alpha_1 \geqslant n(1,2)$	1 never occurs more than once in a given column
$n(1,1)+n(2,2) \geqslant n(1,2)+n(2,2)$	first row $\geqslant$ second row
$n(1,1) \geqslant n(2,2)$	LR condition.

Since the equations (3.3) are homogeneous we conclude from theorem 3.1 that there exist a finite number of 'fundamental' solutions to these equations (the zero solution is excluded) with the property that every other solution may be expressed as a sum of these fundamental solutions and that each fundamental solution is not the sum of any other two (non-zero) solutions. This is expressed in the following proposition.

Proposition 3.4. $\boldsymbol{LR}^m$ has a finite number of elementary tableaux.

Huet (1978) has given an algorithm for generating the fundamental solutions of a single homogeneous linear diophantine equation. This may be easily generalised to the case of simultaneous homogeneous linear diophantine equations. We have written a computer program in Pascal to do this and the results will be presented in the next section.

Returning to our example, we find in this case the fundamental solutions

$$\begin{array}{l}
(\alpha_1, \alpha_2, n(1,1), n(1,2), n(2,2), s_{1,2}, s_{2,2}, t_{2,2}) \\
(1,0,0,0,0,1,1,0) \\
(0,0,1,0,0,0,1,1) \\
(0,1,0,0,0,0,0,0) \\
(0,0,1,0,1,0,0,0) \\
(1,0,0,1,0,0,0,0).
\end{array} \tag{3.4}$$

These correspond to the following elementary tableaux:

$$\begin{array}{|c|}\hline \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline\end{array} \quad \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \quad \begin{array}{|c|}\hline \\ \hline 1 \\ \hline\end{array}\,.$$

It is not difficult to find similar sets of equations for $\boldsymbol{LR}_{\mathrm{o}}^k$ and $\boldsymbol{LR}_{\mathrm{sp}}^k$ and so to deduce that they too have a finite number of elementary tableaux. We omit the details, which are straightforward, but we note that to ensure that a variable x is even it is sufficient to introduce a dummy variable y and the additional equation $x-2y=0$.

Finally we note the following result.

Proposition 3.5. If T_1 and T_2 are two k-Tokuyama tableaux then $T_1 \,.\, T_2$ is a k-Tokuyama tableau.

Proof. Similar to propositions 3.1 and 3.2.

4. Results

In the last section we established that various subsets of $\boldsymbol{T}$ are closed under the stretched product operation and that in some cases the number of elementary tableaux is finite. We shall proceed in this section to find the elementary tableaux and syzygies (algebraic relations) for $\boldsymbol{LR}^4$, $\boldsymbol{LR}^3_{\mathrm{o}}$ and $\boldsymbol{LR}^5_{\mathrm{sp}}$. The theory of Grobner bases (Buchberger 1989) plays an essential role in the calculation of these syzygies and also provides the key to calculating the final generating function.

To illustrate the general situation, let us consider a simple example, $\mathrm{SO}(n) \subset \mathrm{SU}(n)$ with only the first two labels non-zero (this is the example considered in Moshinsky and Devi (1969) section 4B). We assume that n is sufficiently large that theorem 2.2(ii) may be applied (so that in fact we are considering $\boldsymbol{LR}^2_{\mathrm{o}}$), in this case $n \geqslant 5$. Then, for example, the multiplicity of the representation (3) inside the representation (4, 3) is given by the number of LR tableaux of shape $(4,3)-(3)$ such that the weight is even. There is only one possibility:

$$\begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 1 & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array}$$

and so the multiplicity is one. From the results of the last section the tableaux in $\boldsymbol{LR}^2_{\mathrm{o}}$ may be multiplied together using the stetched product. Moreover, the elementary tableaux may be found by finding the fundamental solutions to the following set of homogeneous linear Diophantine equations (this takes only a few seconds of CPU time on a VAX 8550):

$$\begin{aligned}
&-n(1,2)+\alpha_1-s_{1,2}=0\\
&n(1,1)-n(1,2)-n(2,2)+\alpha_1-s_{2,2}=0\\
&n(1,1)-n(2,2)-t_{2,2}=0\\
&n(1,1)+n(1,2)-2u_1=0\\
&n(2,2)-2u_2=0
\end{aligned}\tag{4.1}$$

where u_1 and u_2 are new dummy variables. Note that although the dummy variables $s_{1,2}$, $s_{2,2}$, $t_{2,2}$, u_1 and u_2 are necessary to impose the constraints they are not needed to reconstruct the tableaux since $n(1,1)$, $n(1,2)$, $n(2,2)$, α_1 and α_2 are a complete set of parameters.

Every possible tableau may then be obtained by taking products of the elementary tableaux which may be found in table 4(a) where they have been labelled a, $\bar{a}$, b, $\bar{b}$, e and m. The expression for the above tableau in terms of these is given by:

$$\begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 1 & 1 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3}\end{array} = e.m = \begin{array}{|c|c|}\hline & \\ \hline 1 & 1 \\ \hline\end{array} \cdot \begin{array}{|c|c|}\hline & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

Now suppose we wish to construct a generating function for this branching rule. This will be a rational function of the auxiliary variables a_1, a_2, b_1 and b_2 such that the coefficient of the term $a_1^a a_2^b b_1^p b_2^q$ in the power series expansion is the multiplicity of the $\mathrm{SO}(n)$ representation (p, q) in the $\mathrm{SU}(n)$ representation (a, b) (we use Dynkin labelling in the generating functions. Since the number of non-zero labels is sufficiently small the relationship between the two labellings is given by (2.1)). We can try to find the form of this generating function by using the elementary tableaux.

Consider tableaux obtained from the product $e^x m^y$ of e and m; since e corresponds to a term $a_2^2 b_1^2$ and m corresponds to $a_2 a_1 b_1$ in the expanded generating function, it is not difficult to see that $e^x m^y$ corresponds to a term $(a_2^2 b_1^2)^x (a_2 a_1 b_1)^y$. Summing over all x and y produces a term $(1 - a_2^2 b_1^2)^{-1}(1 - a_2 a_1 b_1)^{-1}$ in the generating function. This might suggest that the full generating function is given by $\Pi_i (1 - l_i)^{-1}$ where l_i are the auxiliary labels of the ith elementary tableau and the product is taken over all the elementary tableaux. Unfortunately things are not quite so simple.

If we calculate the multiplicity of the representation (2) inside (4, 2) we find that it is one since we have only the single tableau

$$\begin{array}{|c|c|c|c|}\hline & & 1 & 1 \\ \hline 1 & 1 \\ \cline{1-2} \end{array} \,. \tag{4.2}$$

On the other hand if we consider the generating function suggested above we find that the multiplicity is two since $a_2^2 a_1^2 b_1^2$ occurs in the expansion of $(1 - a_2 a_1 b_1)^{-1}$ from the elementary multiplet m and in $(1 - a_1^2)^{-1}(1 - a_2^2 b_1^2)^{-1}$ from the elementary multiplets $\bar{a}$ and e. The problem is clear if we construct the stretched products:

$$m^2 = \begin{array}{|c|c|}\hline & 1 \\ \hline 1 \\ \cline{1-1} \end{array} \cdot \begin{array}{|c|c|}\hline & 1 \\ \hline 1 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|}\hline & \\ \hline 1 & 1 \\ \hline \end{array} \cdot \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} = e\bar{a}.$$

We see that the tableau (4.2) may be expressed in two ways as a product of elementary tableaux. In other words, there is a syzygy $m^2 = e\bar{a}$. The effect of this is that $(1 - a_2 a_1 b_1)^{-1}$ must be replaced by $(1 + a_2 a_1 b_1)$, corresponding to the fact that in the general expression for a Young tableau as a product of elementary tableaux we may eliminate m^{2k}. In fact no other changes are necessary and the form of the generating function is:

$$(1 - a_1 b_1)^{-1}(1 - a_1^2)^{-1}(1 - a_2 b_2)^{-1}(1 - a_2^2)^{-1}(1 - a_2^2 b_1^2)^{-1}(1 + a_2 a_1 b_1). \tag{4.3}$$

In order to generalise this example to more difficult cases it is necessary to introduce more machinery. First let us denote by R' the commutative ring with unit generated by the elements of $\boldsymbol{LR}_{\mathrm{o}}^3$ with a unit adjoined and set $R = \mathbb{Q} \otimes_{\mathbb{Z}} R'$. Let $P = \mathbb{Q}[a, \bar{a}, b, \bar{b}, e, m]$ (we consider these letters now to be indeterminates rather than elementary multiplets), then we have a surjection of rings $f: P \to R$ where a, $\bar{a}$, b, $\bar{b}$, e and m map to the corresponding Young tableaux. The problem is that f is not an isomorphism since, for example, $f(m^2 - e\bar{a}) = 0$. Our task then is to find generators for the kernel, K, of f in P, i.e. a basis for the (first-order) syzygies satisfied by the elementary tableaux. To do this it is convenient first to replace R by an isomorphic subring of $S = \mathbb{Q}[x_1, x_2, x_3, x_4, x_5]$ using the parametrisation of $\boldsymbol{LR}_{\mathrm{o}}^3$ given above. Given any tableaux T with parameters $n(1,1)$, $n(1,2)$, $n(2,2)$, α_1, α_2, we define $j: R \to S$ by $j(T) = x_1^{n(1,1)} x_2^{n(1,2)} x_3^{n(2,2)} x_4^{\alpha_1} x_5^{\alpha_2}$ and extend linearly. It is not difficult to see that j is injective and that its image, I, is the subring generated by the images of the elementary tableaux, so $I = \langle x_4, x_1^2, x_5, x_1^2 x_3^2, x_2^2 x_4^2, x_1 x_2 x_4 \rangle$. Now letting $i = j \circ f$ we see that $I \simeq P/K$, so finding generators for K is equivalent to finding generators for the syzygies, or algebraic relations, satisfied by the generators of I. One such relation is $(x_1 x_2 x_4)^2 - (x_1^2)(x_2^2 x_4^2) = 0$ where the left-hand side is the image of $m^2 - e\bar{a}$. This problem can be solved using the theory of Grobner bases. The reader is referred to the review article by Buchberger (1989) for some details. We shall only require the two results given below.

Here $\boldsymbol{F}$ is a finite set of polynomials from $\mathbb{Q}[x_1, \ldots, x_n]$, Ideal($\boldsymbol{F}$) is the ideal generated by the elements of $\boldsymbol{F}$ in $\mathbb{Q}[x_1, \ldots, x_n]$, $\boldsymbol{GB}(\boldsymbol{F})$ is another set of polynomials which also generate Ideal($\boldsymbol{F}$) which is known as a (reduced) Grobner basis of Ideal($\boldsymbol{F}$). $\boldsymbol{GB}(\boldsymbol{F})$ is obtained from $\boldsymbol{F}$ by Buchberger's algorithm (Buchberger 1989, theorem 2.3.1). Note that the calculation of $\boldsymbol{GB}(\boldsymbol{F})$ requires the choice of some 'admissible' ordering for the monomials of $\mathbb{Q}[x_1, \ldots, x_n]$, (Buchberger 1989, p 54), for example lexical, degree-lexical etc. Different orderings will in general lead to different Grobner bases (and different generating functions). $[u]_F$ is the residue class of the polynomial u modulo Ideal($\boldsymbol{F}$) and $\boldsymbol{MLP}(\boldsymbol{F})$ is the set of multiples of leading terms (with respect to the given ordering) in $\boldsymbol{F}$.

Theorem 4.1 (Buchberger 1989, p 61). Let $\boldsymbol{F} = \{f_1, \ldots, f_m\}$ be a set of polynomials from $\mathbb{Q}[x_1, \ldots, x_n]$, let $y_1, \ldots, y_m$ be new indeterminates and let $<$ be the lexical ordering defined by $y_1 < \ldots < y_m < x_1 < \ldots < x_n$. Then $\boldsymbol{GB}(\{y_1 - f_1, \ldots, y_m - f_m\}) \cap \mathbb{Q}[y_1, \ldots, y_m]$ is a reduced Grobner basis for the 'ideal of algebraic relations' of $\boldsymbol{F}$ over $\mathbb{Q}$, i.e. for the set $\{g \in \mathbb{Q}[y_1, \ldots, y_m] | g(f_1, \ldots, f_m) = 0\}$.

Theorem 4.2 (Buchberger 1989, p 58). The set $\{[u]_F | u \notin \boldsymbol{MLP}(\boldsymbol{GB}(\boldsymbol{F}))\}$ is a linearly independent basis for $\mathbb{Q}[x_1, \ldots, x_m]/\text{Ideal}(\boldsymbol{F})$ considered as a vector space over $\mathbb{Q}$.

To see how these two theorems are used let us apply them to our example. Here $\boldsymbol{F} = \{x_4, x_1^2, x_5, x_1^2x_3^2, x_2^2x_4^2, x_1x_2x_4\}$ and since $\boldsymbol{F}$ has six elements we introduce six new variables y_1, y_2, y_3, y_4, y_5 and y_6 which we shall identify with a, $\bar{a}$, b, $\bar{b}$, e and m. We take the lexical ordering with $x_1 > x_2 > x_3 > x_4 > x_5 > m > e > \bar{b} > b > \bar{a} > a$. To find the algebraic relations satisfied by the elements of $\boldsymbol{F}$, we apply the first theorem, i.e. we apply Buchberger's algorithm to the set of polynomials $\{x_4 - a, x_1^2 - \bar{a}, x_5 - b, x_1^2x_3^2 - \bar{b}, x_2^2x_4^2 - e, x_1x_2x_4 - m\}$ which are considered to be generators of an ideal in $\mathbb{Q}[x_1, \ldots, x_5, a, \ldots, m]$. All such calculations have been done using either the Grobner basis package in MAPLE† or using the program CoCoA‡. The required Grobner basis here is $\{x_1^2 - \bar{a}, x_1x_2a - m, x_1m - x_2a\bar{a}, x_1e - x_2ma, x_2^2a^2 - e, x_3^2\bar{a} - \bar{b}, x_4 - a, x_5 - b, m^2 - e\bar{a}\}$. The intersection of this basis with $P = \mathbb{Q}[a, \ldots, m]$ is just $\{m^2 - e\bar{a}\}$ which is thus, from the first theorem, the generator for the ideal K introduced above. Thus there is essentially only one syzygy satisfied by the elementary multiplets. Now P/K is isomorphic to R by construction. On the other hand, the second theorem tells us how to construct a vector space basis for P/K. It consists of the residue classes of all the monomials that are not multiples of the leading term of $m^2 - e\bar{a}$. Since $m > e > \bar{a}$ the leading term is simply m^2, and so the monomials which are not multiples of m^2 are simply all monomials in a, $\bar{a}$, b, $\bar{b}$, e and m which contain m linearly or not at all, with no restrictions on the other variables. This basis of P/K may be identified with the elements $\boldsymbol{LR}_0^3$ and so we have constructed a one-to-one correspondence between the tableaux of $\boldsymbol{LR}_0^3$ and restricted products of elementary tableaux. It is now a simple matter to construct the required generating function. We grade P by associating with each of the generators a, $\bar{a}$, b, $\bar{b}$, e and m the group-subgroup labels of the corresponding elementary multiplet. This induces a grading on P/K since K is homogeneous with respect to this grading. Since we have a basis of P/K consisting of products of (the images of) the generators of P it is not difficult to write down a generating function

† MAPLE is a system for algebraic computing: see Char B W, Geddes K O, Gaston H G, Monaghan H B and Watt S M 1988 *The Maple Reference Manual* (5th edition) (Waterloo: Watcom).

‡ CoCoA is a system for COmputations in COmmutative Algebra which runs on any Macintosh with at least 512Kb of RAM. It has been written by Alessandro Giovini and Giangranco Niesi (1989) and is available free of charge by sending a blank diskette to: Alessandro Giovini or Gianfranco Niesi, Department of Mathematics, University of Genova, viale Leon Battista Alberti 4, 16132, Genova, Italy.

for the dimensions of each graded subspace. Denominator terms come from 'unrestricted' variables while numerator terms come from 'restricted' variables or from a need to avoid overcounting. This is a generalisation of the calculation of a generating function for the Poincaré series.

In general we proceed in the same way. First we find the elementary tableaux, then apply theorem 4.1 to find a Grobner basis for the algebraic relations between these elementary tableaux. Finally a decomposition of tableaux into products of elementary tableaux is given by writing down the general forms of the monomials which are not multiples of the leading terms of the elements of the Grobner basis we have found (in the previous literature this was known as eliminating forbidden products). Finally we write down the generating function.

Consider now the branching $\mathrm{SU}(n)\times\mathrm{SU}(m)\subset\mathrm{SU}(n+m)$ with the first four Dynkin labels non-zero. The elementary tableaux of $\boldsymbol{LR}^4$ are required and these are given in table 2(a) together with their auxiliary labels. Each elementary tableau is also labelled by a single letter or barred letter for notational simplicity. The bars are used to emphasise the symmetry under the exchange of the two subgroups $\mathrm{SU}(n)$ and $\mathrm{SU}(m)$.

We find that the syzygies all take the form $xy=\ldots$ where x and y are two elementary tableaux, so they may be presented in the form of a 'compatibility table' which is given in table 2(b) and the corresponding syzygies are in table 2(c). These lead to the following possible expression for the branching rule generating function in which $Z=(1-l_z)^{-1}$, l_z being the auxiliary labels of the elementary multiplet z as recorded in table 2(a). These auxiliary labels may be obtained from the corresponding Young tableaux by first using theorem 2.2 to find the labels of the associated $\mathrm{SU}(n+m)$ and $\mathrm{SU}(n)\times\mathrm{SU}(m)$ representations. These are then converted to Dynkin labels using (2.1). So, for example, the elementary multiplet $\bar{l}$ is associated with an $\mathrm{SU}(n+m)$ representation with partition labels (2, 2, 1, 1) and Dynkin labels (0, 1, 0, 1) given by all the boxes of the diagram, an $\mathrm{SU}(n)$ representation with partition and Dynkin labels (2) given by the empty boxes in the diagram and an $\mathrm{SU}(m)$ representation with partition labels (2, 1, 1) and Dynkin labels (1, 0, 1) given by arranging the numbered boxes into a Young diagram. Finally the auxiliary variables carry these Dynkin labels as exponents. The generating function is

$$\begin{aligned} G_1(a_i, b_i, c_i) = {} & ABCD\overline{ABCD}G\bar{G}H(FE\bar{F}+lE\bar{F}L+j\bar{F}LJ+kLJK \\ & +\bar{j}JK\bar{J}+\bar{l}K\bar{J}\bar{L}+f\bar{J}\bar{L}F+\bar{l}e\bar{L}FE+iF\bar{F}I+i\bar{j}F\bar{J}I \\ & +ji\bar{F}JI+ij\bar{j}J\bar{J}I+keKEL+ke\bar{l}KE\bar{L}). \end{aligned} \tag{4.4}$$

Similarly, the elementary tableaux, compatibility table and syzygies for the rank-free branching $\mathrm{Sp}(2n)\subset\mathrm{SU}(2n)$ with five non-zero labels are given in tables 3(a), 3(b) and 3(c). They yield the following generating function:

$$\begin{aligned} G_2(a_i, b_i, c_i) = {} & ABCEFHIJKLO(DGMN+rDGMR+pDGNP \\ & +qDGPQ+qrDGQR+sDMNS+rsDMRS+tDNPT \\ & +stDNST+qtDPQT+xDQRX+txDQTX+sxDRSX \\ & +stxDSTX+vGMNV+uGMRU+uvGMUV+pvGNPV \\ & +wGPQW+vwGPVW+quGQRU+uwGQUW+uvwGUVW \\ & +svMNSV+suMRSU+suvMSUV+tvNPTV+stvNSTV \\ & +twPQTW+tvwPTVW+uxQPUX+tuQTUW+tuxQTUX \\ & +suxRSUX+stuSTUX+tuvSTUV+tuvwTUVW). \end{aligned} \tag{4.5}$$

Table 2(*a*). Elementary multiplets for $SU(n)\times SU(m)\subset SU(n+m)$, four labels.

Multiplet	Auxiliary labels	Tableau	Multiplet	Auxiliary labels	Tableau
a	a_1b_1	$\boxed{}$	$\bar{g}$	$a_4b_1c_3$	$\begin{array}{l}\boxed{}\\\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}$
$\bar{a}$	a_1c_1	$\boxed{1}$	h	$a_4b_2c_2$	$\begin{array}{l}\boxed{}\\\boxed{}\\\boxed{1}\\\boxed{2}\end{array}$
b	a_2b_2	$\begin{array}{l}\boxed{}\\\boxed{}\end{array}$	i	$a_3a_1b_2c_2$	$\begin{array}{ll}\boxed{}&\boxed{1}\\\boxed{}\\\boxed{2}\end{array}$
$\bar{b}$	a_2c_2	$\begin{array}{l}\boxed{1}\\\boxed{2}\end{array}$	j	$a_4a_1b_3c_2$	$\begin{array}{ll}\boxed{}&\boxed{1}\\\boxed{}\\\boxed{}\\\boxed{2}\end{array}$
c	a_3b_3	$\begin{array}{l}\boxed{}\\\boxed{}\\\boxed{}\end{array}$	$\bar{j}$	$a_4a_1b_2c_3$	$\begin{array}{ll}\boxed{}&\boxed{1}\\\boxed{}\\\boxed{2}\\\boxed{3}\end{array}$
$\bar{c}$	a_3c_3	$\begin{array}{l}\boxed{1}\\\boxed{2}\\\boxed{3}\end{array}$	k	$a_4a_2b_3c_3$	$\begin{array}{ll}\boxed{}&\boxed{1}\\\boxed{}&\boxed{2}\\\boxed{}\\\boxed{3}\end{array}$
d	a_4b_4	$\begin{array}{l}\boxed{}\\\boxed{}\\\boxed{}\\\boxed{}\end{array}$	l	$a_4a_2b_3b_1c_2$	$\begin{array}{ll}\boxed{}&\boxed{}\\\boxed{}&\boxed{1}\\\boxed{}\\\boxed{2}\end{array}$
$\bar{d}$	a_4c_4	$\begin{array}{l}\boxed{1}\\\boxed{2}\\\boxed{3}\\\boxed{4}\end{array}$	$\bar{l}$	$a_4a_2b_2c_3c_1$	$\begin{array}{ll}\boxed{}&\boxed{1}\\\boxed{}&\boxed{2}\\\boxed{1}\\\boxed{3}\end{array}$
e	$a_2b_1c_1$	$\begin{array}{l}\boxed{}\\\boxed{1}\end{array}$			
f	$a_3b_2c_1$	$\begin{array}{l}\boxed{}\\\boxed{}\\\boxed{1}\end{array}$			
$\bar{f}$	$a_3b_1c_2$	$\begin{array}{l}\boxed{}\\\boxed{1}\\\boxed{2}\end{array}$			
g	$a_4b_3c_1$	$\begin{array}{l}\boxed{}\\\boxed{}\\\boxed{}\\\boxed{1}\end{array}$			

Table 2(*b*). Compatibility table for $SU(n) \times SU(m) \subset SU(n+m)$.

	e	f	$\bar{f}$	i	j	$\bar{j}$	k	l	$\bar{l}$
e	√	√	√	1	2	3	√	√	√
f		√	√	√	4	√	5	6	√
$\bar{f}$			√	√	√	7	8	√	9
i				√	√	√	10	11	12
j					√	√	√	√	13
$\bar{j}$						√	√	14	√
k							√	√	√
l								√	15
$\bar{l}$									√

Table 2(*c*). Syzygies for $SU(n) \times SU(m) \subset SU(n+m)$.

1	$ei = \bar{a}b\bar{f}$	9	$\bar{f}\bar{l} = \bar{b}f\bar{g}$
2	$ej = \bar{a}l$	10	$ik = \bar{b}c\bar{j}$
3	$e\bar{j} = \bar{a}b\bar{g}$	11	$il = b\bar{f}j$
4	$fj = \bar{a}ch$	12	$i\bar{l} = \bar{b}f\bar{j}$
5	$fk = c\bar{l}$	13	$j\bar{l} = \bar{a}hk$
6	$fl = ceh$	14	$\bar{j}l = b\bar{g}j$
7	$\bar{f}\bar{j} = \bar{g}i$	15	$l\bar{l} = ehk$
8	$\bar{f}k = \bar{b}c\bar{g}$		

Ordering: deglex

lex:	l	$\bar{l}$	k	j	$\bar{j}$	i	f	$\bar{f}$	e	a	b	c	d	g	h	$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{d}$	$\bar{g}$
deg:	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0

Finally the results for the branching $SO(n) \subset SU(n)$ with three non-zero labels are contained in tables 4(*a*), 4(*b*) and 4(*c*) and yield:

$$G_3(a_i, b_i, c_i) = ABC\overline{ABC}F\bar{F}[(1+p)(1+m+n+o)E + (1+o)(i+q+r+qr)I]. \quad (4.6)$$

Note that we can deduce the 'non-numerator' part of the generating function (4.6) from the generating function for $SU(n) \times SU(m) \subset SU(n+m)$ with three non-zero labels. This corresponds to doubling all the rows in the elementary tableaux except for those with no integers in them. We have labelled the elementary tableaux according to this correspondence. The numerator terms are elementary tableaux which do not arise in this way since some of their rows contain single entries. Their squares, however, do have doubled entries and so are expressible in terms of the denominator terms.

A simple check can be applied to these generating functions: the number of denominator factors in each term should be equal to the number of group labels, representation plus internal, minus the number of internal subgroup labels (Seligman and Sharp 1980). For $SU(m) \times SU(n) \subset SU(m+n)$, first four labels non-zero, we have $4m+4n-6$ as the number of group labels, $4m+4n-20$ as the number of internal subgroup labels. The difference, 14, is the number of denominator factors.

For $Sp(2n) \subset SU(2n)$, first five labels non-zero, we have $10n-10$ as the number of group labels, $10n-25$ as the number of internal subgroup labels and hence 15 as the number of denominator factors. For $SO(n) \subset SU(n)$, first three labels non-zero, we have $3n-3$ group labels, $3n-12$ internal subgroup labels, and hence 9 denominator factors.

In each case the number of denominator factors is rank-independent and agrees with the generating functions given above.

Table 3(*a*). Elementary multiplets for $\mathrm{Sp}(2n)\subset \mathrm{SU}(2n)$, five labels.

Multiplet	Auxiliary labels	Tableau
a	a_1b_1	$\boxed{}$
b	a_2b_2	$\begin{array}{l}\boxed{}\\ \boxed{}\end{array}$
c	a_3b_3	$\begin{array}{l}\boxed{}\\ \boxed{}\\ \boxed{}\end{array}$
d	a_4b_4	$\begin{array}{l}\boxed{}\\ \boxed{}\\ \boxed{}\\ \boxed{}\end{array}$
e	a_5b_5	$\begin{array}{l}\boxed{}\\ \boxed{}\\ \boxed{}\\ \boxed{}\\ \boxed{}\end{array}$
f	a_2	$\begin{array}{l}\boxed{1}\\ \boxed{2}\end{array}$
g	a_3b_1	$\begin{array}{l}\boxed{}\\ \boxed{1}\\ \boxed{2}\end{array}$
h	a_4b_2	$\begin{array}{l}\boxed{}\\ \boxed{}\\ \boxed{1}\\ \boxed{2}\end{array}$
i	a_5b_3	$\begin{array}{l}\boxed{}\\ \boxed{}\\ \boxed{}\\ \boxed{1}\\ \boxed{2}\end{array}$
j	a_4	$\begin{array}{l}\boxed{1}\\ \boxed{2}\\ \boxed{3}\\ \boxed{4}\end{array}$
k	a_5b_1	$\begin{array}{l}\boxed{}\\ \boxed{1}\\ \boxed{2}\\ \boxed{3}\\ \boxed{4}\end{array}$
l	$a_5a_3b_4$	$\begin{array}{ll}\boxed{}&\boxed{1}\\ \boxed{}&\boxed{2}\\ \boxed{}&\boxed{3}\\ \boxed{}&\\ \boxed{4}&\end{array}$
m	$a_3a_1b_2$	$\begin{array}{ll}\boxed{}&\boxed{1}\\ \boxed{}&\\ \boxed{2}&\end{array}$
n	$a_4a_1b_3$	$\begin{array}{ll}\boxed{}&\boxed{1}\\ \boxed{}&\\ \boxed{}&\\ \boxed{2}&\end{array}$
o	$a_5a_1b_4$	$\begin{array}{ll}\boxed{}&\boxed{1}\\ \boxed{}&\\ \boxed{}&\\ \boxed{}&\\ \boxed{2}&\end{array}$
p	$a_4a_2b_3b_1$	$\begin{array}{ll}\boxed{}&\boxed{}\\ \boxed{}&\boxed{1}\\ \boxed{}&\\ \boxed{2}&\end{array}$
q	$a_5a_2b_4b_1$	$\begin{array}{ll}\boxed{}&\boxed{}\\ \boxed{}&\boxed{1}\\ \boxed{}&\\ \boxed{}&\\ \boxed{2}&\end{array}$
r	$a_5a_3b_4b_2$	$\begin{array}{ll}\boxed{}&\boxed{}\\ \boxed{}&\boxed{}\\ \boxed{}&\boxed{1}\\ \boxed{}&\\ \boxed{2}&\end{array}$
s	$a_5a_1b_2$	$\begin{array}{ll}\boxed{}&\boxed{1}\\ \boxed{}&\\ \boxed{2}&\\ \boxed{3}&\\ \boxed{4}&\end{array}$
t	$a_5a_2b_3$	$\begin{array}{ll}\boxed{}&\boxed{1}\\ \boxed{}&\boxed{2}\\ \boxed{}&\\ \boxed{3}&\\ \boxed{4}&\end{array}$

Table 3(*a*). (continued)

Multiplet	Auxiliary labels	Tableau	Multiplet	Auxiliary labels	Tableau
u	$a_5a_3b_2$	$\begin{array}{ll} \square & 1 \\ \square & 2 \\ 1 & 3 \\ 2 & \\ 4 & \end{array}$	w	$a_5a_3a_2b_3b_1$	$\begin{array}{lll} \square & \square & 1 \\ \square & 1 & 2 \\ \square & 3 & \\ 2 & & \\ 4 & & \end{array}$
v	$a_5a_3a_1b_3$	$\begin{array}{lll} \square & 1 & 1 \\ \square & 2 & \\ \square & 3 & \\ 2 & & \\ 4 & & \end{array}$	x	$a_5^2a_2b_4b_2$	$\begin{array}{lll} \square & \square & 1 \\ \square & \square & 2 \\ \square & 1 & \\ \square & 3 & \\ 2 & 4 & \end{array}$

Table 3(*b*). Compatibility table for $Sp(2n) \subset SU(2n)$.

	d	g	m	n	p	q	r	s	t	u	v	w	x
d	√	√	√	√	√	√	√	√	√	1	2	3	√
g		√	√	√	√	√	√	4	5	√	√	√	24
m			√	√	6	7	√	√	8	√	√	9	23
n				√	√	10	11	√	√	12	√	13	25
p					√	√	14	15	√	16	13	√	26
q						√	√	17	√	√	18	√	√
r							√	√	22	√	19	20	√
s								√	√	√	√	21	√
t									√	√	√	√	√
u										√	√	√	27
v											√	√	28
w												√	√
x													√

Table 3(*c*). Syzygies for $Sp(2n) \subset SU(2n)$.

1	$du = lh$	15	$ps = bkn$
2	$dv = ln$	16	$pu = wh$
3	$dw = lp$	17	$qs = bko$
4	$gs = km$	18	$qv = wo$
5	$gt = cfk$	19	$rv = cou$
6	$mp = bgn$	20	$rw = cqu$
7	$mq = bgo$	21	$sw = vbk$
8	$mt = cfs$	22	$rt = xc$
9	$mw = vgb$	23	$mx = rfs$
10	$nq = op$	24	$gx = krf$
11	$nr = cho$	25	$nx = tho$
12	$nu = vh$	26	$px = htq$
13	$nw = vp$	27	$vx = uto$
14	$pr = chq$	28	$wx = utq$

Ordering: deglex

lex:	w	m	r	x	d	g	n	p	q	s	u	v	t	b	c	f	h	k	l	o
deg:	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0

Table 4(*a*). Elementary multiplets for SO(n)$\subset$SU(n), three labels.

Multiplet	Auxiliary labels	Tableau	Multiplet	Auxiliary labels	Tableau
a	a_1b_1	$\begin{ytableau} \none \end{ytableau}$	i	$a_3^2a_1^2b_2^2$	$\begin{ytableau} \none & \none & 1 & 1 \\ \none & \none \\ 2 & 2 \end{ytableau}$
$\bar{a}$	a_1^2	$\begin{ytableau} 1 & 1 \end{ytableau}$	m	$a_2a_1b_1$	$\begin{ytableau} \none & 1 \\ 1 \end{ytableau}$
b	a_2b_2	$\begin{ytableau} \none \\ \none \end{ytableau}$	n	$a_3a_2b_2b_1$	$\begin{ytableau} \none & \none \\ \none & 1 \\ 1 \end{ytableau}$
$\bar{b}$	a_2^2	$\begin{ytableau} 1 & 1 \\ 2 & 2 \end{ytableau}$	o	$a_3a_1b_2$	$\begin{ytableau} \none & 1 \\ \none \\ 1 \end{ytableau}$
c	a_3b_3	$\begin{ytableau} \none \\ \none \\ \none \end{ytableau}$	p	$a_3a_2b_1$	$\begin{ytableau} \none & 1 \\ 1 & 2 \\ 2 \end{ytableau}$
$\bar{c}$	a_3^2	$\begin{ytableau} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{ytableau}$	q	$a_3^2a_1b_2b_1$	$\begin{ytableau} \none & \none & 1 \\ \none & 1 \\ 2 & 2 \end{ytableau}$
e	$a_2^2b_1^2$	$\begin{ytableau} \none & \none \\ 1 & 1 \end{ytableau}$	r	$a_3a_2a_1b_2$	$\begin{ytableau} \none & 1 & 1 \\ \none & 2 \\ 2 \end{ytableau}$
f	$a_3^2b_2^2$	$\begin{ytableau} \none & \none \\ \none & \none \\ 1 & 1 \end{ytableau}$			
$\bar{f}$	$a_3^2b_1^2$	$\begin{ytableau} \none & \none \\ 1 & 1 \\ 2 & 2 \end{ytableau}$			

Table 4(*b*). Compatibility table for SO(n)$\subset$SU(n).

	e	i	m	n	o	p	q	r
e	√	1	√	√	2	√	3	4
i		√	5	6	√	7	√	√
m			8	2	9	√	10	11
n				12	13	√	14	15
o					16	√	√	√
p						17	18	19
q							20	7
r								21

Table 4(c). Syzygies for $SO(n) \subset SU(n)$.

1	$ei = b^2\bar{a}\bar{f}$	12	$n^2 = fe$
2	$eo = mn$	13	$no = fm$
3	$eq = b\bar{f}m$	14	$nq = b\bar{f}o$
4	$er = bmp$	15	$nr = bop$
5	$mi = \bar{a}bq$	16	$o^2 = \bar{a}f$
6	$ni = boq$	17	$p^2 = \bar{b}\bar{f}$
7	$pi = qr$	18	$pq = \bar{f}r$
8	$m^2 = \bar{a}e$	19	$pr = \bar{b}q$
9	$mo = \bar{a}n$	20	$q^2 = \bar{f}i$
10	$mq = \bar{a}b\bar{f}$	21	$r^2 = \bar{b}i$
11	$mr = \bar{a}bp$		

Ordering: deglex

lex:	e	i	m	n	o	p	q	r	a	b	c	f	$\bar{a}$	$\bar{b}$	$\bar{c}$	$\bar{f}$
deg:	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0

5. Other generating functions

In the last section we gave some new generating functions for rank-free branching rules using the Young tableaux method. In this section we give some further examples of problems that might be attacked.

5.1. SU(3) generating functions

The elementary multiplets for character generators correspond to the weight spaces of the fundamental representations of the group (Stanley 1980, Baclawski 1983, King and El-Sharkaway 1983, 1984). In terms of Young tableaux for the case of SU(3) these are the 3-unitary tableaux which have a single column of length 1 or 2 (see table 5).

The problem is how to find the syzygies and hence the generating function. Baclawski has desribed how these may be found for $SU(n)$ and $Sp(2n)$ by introducing a partial ordering on the elementary tableaux whereby $x < y$ if and only if the juxtaposition xy is a standard tableau; King and El-Sharkaway have extended this procedure to the orthogonal groups and have also described a convenient realisation of the partial

Table 5. Elementary multiplets for SU(3) character generator.

Multiplet	Auxiliary labels	Tableau	Multiplet	Auxiliary labels	Tableau
a	A_1x_1	$\boxed{1}$	d	A_2x_2	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$
b	$A_1\bar{x}_1x_2$	$\boxed{2}$	e	$A_2x_1\bar{x}_2$	$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array}$
c	$A_1\bar{x}_2$	$\boxed{3}$	f	$A_2\bar{x}_1$	$\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline\end{array}$

ordering as a graph whose nodes lie in $\mathbb{R}^n$. For SU(3) this graph takes the form:

$$\begin{tikzcd} 1 \arrow[d] & 2 \arrow[l] \arrow[d] & 3 \arrow[l] \\ \genfrac{}{}{0pt}{}{1}{3} \arrow[d] & \genfrac{}{}{0pt}{}{2}{3} \arrow[l] & \\ \genfrac{}{}{0pt}{}{1}{2} & & \end{tikzcd} \tag{5.1}$$

where the partial ordering is such that $x<y$ if and only if we may link y to x by a directed path. Thus $\genfrac{}{}{0pt}{}{1}{3}<3$ which corresponds to the fact that the juxtaposition $\genfrac{}{}{0pt}{}{1\,3}{3}$ is standard, but 1 and $\genfrac{}{}{0pt}{}{2}{3}$ are not comparable. Now the generating function is constructed by finding the maximal chains c and their decent sets $d(c)$ (for definitions see Stanley (1980), King and El-Sharkaway (1983, 1984), Baclawski (1983)) and then the generating function is given by the expression:

$$\sum_c \left[\prod_{x_j \in d(c)} l(x_j) \left(\prod_{x_j \in c} (1 - l(x_j)) \right)^{-1} \right]. \tag{5.2}$$

Here $l(x_j)$ are the auxiliary labels associated with x_j as given in table 5. There are two maximal chains for SU(3), namely $\{\genfrac{}{}{0pt}{}{1}{2}, \genfrac{}{}{0pt}{}{1}{3}, 1, 2, 3\}$ and $\{\genfrac{}{}{0pt}{}{1}{2}, \genfrac{}{}{0pt}{}{1}{3}, \genfrac{}{}{0pt}{}{2}{3}, 2, 3\}$ with the second having decent set $\{\genfrac{}{}{0pt}{}{2}{3}\}$. Thus the generating function is

$$\frac{1}{(1-A_1\bar{x}_2)(1-A_1\bar{x}_1x_2)(1-A_2x_1\bar{x}_2)(1-A_2x_2)}\left(\frac{1}{(1-A_1x_1)}+\frac{A_2\bar{x}_1}{(1-A_2\bar{x}_1)}\right). \tag{5.3}$$

The fact that the product of 1 and $\genfrac{}{}{0pt}{}{2}{3}$ is forbidden corresponds to the fact that they are not comparable in the partial ordering. This incompatibility can be understood in terms of the stretched product from the fact that there is a syzygy $e.b=f.a$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \cdot \boxed{2} = \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array} \cdot \boxed{1} \; . \tag{5.4}$$

In this case things are quite simple since the graph (5.1) provides a convenient way of coding the syzygies satisfied by the elementary multiplets and hence of constructing the generating function. It would be interesting if similar constructions could be found in other cases. An example is provided by the orbit generator of SU(3). In terms of tableaux we now consider the subset of 3-unitary tableaux which have dominant SU(3) weight (i.e. all weight components non-negative). It is clear that this subset is closed under the stretched product and we may once again look for elementary tableaux. These are contained in table 6 and the syzygies are in table 7, we are led to the following generating function:

$$AB(CGE+fGEF+hEFH+dFHD) \tag{5.5}$$

which is in agreement with that given in Michel *et al* (1988).

Table 6. Elementary multiplets for the SU(3) orbit generator.

Multiplet	Auxiliary labels	Tableau	Multiplet	Auxiliary labels	Tableau
a	A_1A_2	$\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$	*e*	A_1x_1	$\begin{ytableau} 1 \end{ytableau}$
b	A_1A_2	$\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}$	*f*	A_2x_2	$\begin{ytableau} 1 \\ 2 \end{ytableau}$
c	A_1^3	$\begin{ytableau} 1 & 2 & 3 \end{ytableau}$	*g*	$A_1^2x_2$	$\begin{ytableau} 1 & 2 \end{ytableau}$
d	A_2^3	$\begin{ytableau} 1 & 1 & 2 \\ 2 & 3 & 3 \end{ytableau}$	*h*	$A_2^2x_1$	$\begin{ytableau} 1 & 1 \\ 2 & 3 \end{ytableau}$

Table 7. Syzygies for SU(3) orbit generator.

1	$gh = aef$	4	$de = ah$
2	$cf = bg$	5	$dg = a^2f$
3	$ch = abe$	6	$cd = a^2b$

The generating function for the multiplicities of tensors in the enveloping algebra of SU(3) can also be reproduced. There are five elementary tableaux as shown in table 8, together with one syzygy:

$$\begin{aligned} &\begin{ytableau} \; & \; & 1 & 1 \\ \; \\ 2 \end{ytableau} \cdot \begin{ytableau} \; & \; & 1 \\ \; & 1 & 2 \end{ytableau} = \begin{ytableau} \; & \; & \; & \; & 1 & 1 & 1 \\ \; & \; & 1 & 2 \\ 2 \end{ytableau} \\ &\qquad = \begin{ytableau} \; \\ 1 \\ 2 \end{ytableau} \cdot \begin{ytableau} \; & 1 \\ 2 \end{ytableau} \cdot \begin{ytableau} \; & 1 \\ \; \end{ytableau} \cdot \begin{ytableau} \; & 1 \\ \; \end{ytableau} \end{aligned} \tag{5.6}$$

Thus, after eliminating the scalar, we obtain the generating function

$$\begin{aligned} &(1-U^2)^{-1}(1-U^3)^{-1}(1-Ua_1a_2)^{-1}(1-U^2a_1a_2)^{-1} \\ &\qquad \times [(1-U^3a_1^3)^{-1} + U^3a_2^3(1-U^3a_2^3)^{-1}] \end{aligned} \tag{5.7}$$

Table 8. Elementary multiplets for the SU(3) enveloping algebra.

Multiplet	Auxiliary labels	Tableau	Multiplet	Auxiliary labels	Tableau
a	UA_1A_2	$\begin{ytableau} \; & 1 \\ 2 \end{ytableau}$	*d*	U^3	$\begin{ytableau} \; \\ \; \\ \; \end{ytableau}$
b	U^2	$\begin{ytableau} \; \\ \; \\ 1 \end{ytableau}$	*e*	$U^3A_1^3$	$\begin{ytableau} \; & \; & 1 & 1 \\ \; \\ 2 \end{ytableau}$
c	$U^2A_1A_2$	$\begin{ytableau} \; & 1 \\ \; \end{ytableau}$	*f*	$U^3A_2^3$	$\begin{ytableau} \; & \; & 1 \\ \; & 1 & 2 \end{ytableau}$

which is in agreement with Couture and Sharp (1980). Here the exponent of U carries the degree and the exponents of a_1 and a_2 the SU(3) (Dynkin) representation labels.

5.2. Sp(4) generating functions

The Sp(4) character generator can be rederived in much the same way as for SU(3) with the elementary multiplets and syzygies shown in tables 9 and 10. Once again the forbidden products correspond to the elimination of elementary multiplets that are not related by the partial ordering. However, there seems to be some difficulty in

Table 9. Elementary multiplets for the Sp(4) character generator.

Multiplet	Auxiliary labels	Tableau	Multiplet	Auxiliary labels	Tableau
a	$A_1\bar{x}_1$	$\boxed{\bar{1}}$	f	$A_2\bar{x}_1x_2$	$\begin{array}{\|c\|}\hline \bar{1}\\ \hline 2\\ \hline\end{array}$
b	A_1x_1	$\boxed{1}$	g	$A_2x_1\bar{x}_2$	$\begin{array}{\|c\|}\hline 1\\ \hline \bar{2}\\ \hline\end{array}$
c	A_1x_2	$\boxed{2}$			
d	$A_1\bar{x}_2$	$\boxed{\bar{2}}$	h	$A_2x_1x_2$	$\begin{array}{\|c\|}\hline 1\\ \hline 2\\ \hline\end{array}$
e	$A_2\bar{x}_1\bar{x}_2$	$\begin{array}{\|c\|}\hline \bar{1}\\ \hline \bar{2}\\ \hline\end{array}$	i	A_2	$\begin{array}{\|c\|}\hline \bar{2}\\ \hline 2\\ \hline\end{array}$

Table 10. Syzygies for Sp(4) character generator.

1	$ag = be$	4	$bi = dh$
2	$ah = bf$	5	$eh = fg$
3	$ai = df$		

obtaining the simplest form of the orbit generator for Sp(4) by this method. This generating function has been given in Michel *et al* (1988) and takes the form

$$\frac{1}{(1-Q)(1-QB)(1-P^2)(1-PA)} \times\left(\frac{1+PQA}{(1-Q^2A^2)(1-Q^2)}+\frac{P^2B}{(1-P^2)(1-P^2B)}+\frac{P^2}{(1-P^2)(1-Q)}\right) \tag{5.8}$$

where P and Q carry the Sp(4) representation labels (p, q), while A and B carry the orbit labels (a, b). There appear to be at least three elementary tableaux which do not appear in this generating function, namely

$$\begin{array}{|c|c|c|}\hline \bar{1} & 1 & 2\\ \hline \bar{2} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2\\ \hline \bar{2} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline \bar{1} & 1 & 2 & 2\\ \hline \bar{2} & \bar{2} \\ \cline{1-2}\end{array}$$

with corresponding auxiliary labels P^2Q, P^2QB and P^2Q^2, respectively. There are also a large number of syzygies that do not reflect the simplicity of the generating function above. This suggests that using Young tableaux in this case is inefficient; extra elementary multiplets are compensated for by extra syzygies that simplify the final result.

5.3. Generating functions from Tokuyama tableaux

The generating function for the branching $\mathrm{Sp}(4) \subset \mathrm{SU}(4)$ (or $\mathrm{SO}(5) \subset \mathrm{SO}(6)$) is given by

$$\frac{1}{(1-A_1B_1)(1-A_2B_2)(1-A_2)(1-A_3B_1)(1-A_1A_3B_2)(1-A_4)} \tag{5.9}$$

where A_1, A_2 and A_3 carry the three SU(4) representation labels and B_1, B_2 carry the two Sp(4) representation labels. We find the following correspondence between elementary 2-Tokuyama tableaux and the elementary multiplets implicit in this generating function:

$$\begin{array}{lll}
\boxed{1} \sim A_1B_1 & \begin{array}{c} 1 \\ 2 \end{array} \sim A_2B_2 & \begin{array}{c} 1 \\ \bar{1} \end{array} \sim A_2 \\[2ex]
\begin{array}{c} 1 \\ 2 \\ \bar{2} \end{array} \sim A_3B_1 & \begin{array}{cc} 1 & 1 \\ 2 & \\ \bar{1} & \end{array} \sim A_1A_3B_2 & \begin{array}{c} 1 \\ 2 \\ \bar{2} \\ \bar{1} \end{array} \sim A_4 .
\end{array} \tag{5.10}$$

The generating function for $\mathrm{Sp}(6) \subset \mathrm{SU}(6)$ is contained in Couture and Sharp (1980); there are 15 elementary multiplets which include the six found above. Once more there is a correspondence with Tokuyama tableaux, given by

$$\begin{array}{lllll}
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \sim A_3B_3 & \begin{array}{c} 1 \\ 2 \\ 3 \\ \bar{3} \end{array} \sim A_4B_2 & \begin{array}{cc} 1 & 1 \\ 2 & \\ 3 & \\ \bar{1} & \end{array} \sim A_4A_1B_3 & \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & \\ \bar{2} & \end{array} \sim A_4A_2B_1B_3 & \begin{array}{c} 1 \\ 2 \\ 3 \\ \bar{3} \\ \bar{2} \end{array} \sim A_5B_1 \\[2ex]
\begin{array}{cc} 1 & 1 \\ 2 & \\ 3 & \\ \bar{3} & \\ \bar{1} & \end{array} \sim A_5A_1B_2 & \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & \\ \bar{2} & \\ \bar{1} & \end{array} \sim A_5A_2B_3 & \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & \bar{2} \\ \bar{3} & \\ \bar{1} & \end{array} \sim A_5A_3B_2 & \begin{array}{ccc} 1 & 1 & 1 \\ 2 & 2 & \\ 3 & \bar{1} & \\ \bar{2} & & \\ \bar{1} & & \end{array} \sim A_5A_3A_1B_3 . &
\end{array} \tag{5.11}$$

An example of a syzygy is given by the following product:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \bar{3} \\ \cline{1-1} \bar{1} \\ \cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \bar{2} \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline 3 & \bar{2} \\ \hline \bar{3} \\ \cline{1-1} \bar{1} \\ \cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 1 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & \bar{2} \\ \cline{1-2} \bar{3} \\ \cline{1-1} \bar{1} \\ \cline{1-1}\end{array}$$

6. Comments

We conclude with some comments.

(I) We have considered the weight generators and orbit generators for SU(n) and Sp($2n$). King and El-Sharkaway (1983) have also introduced Young tableaux for SO(n) and thereby found weight generating functions. It seems to be difficult, however, to extend the notion of stretched products to these tableaux. A simple example will illustrate the problem. The tableaux for SO(3) consist of a single row with all boxes filled with entries from the set $\{\bar{1}, 1, 0\}$, weakly increasing under the ordering $\bar{1} < 1 < 0$. In this case, however, there is an additional type of constraint, namely that no 1 may occur on the right of a $\bar{1}$. Thus the entries are either a string of 1s followed by 0s or a string of $\bar{1}$s followed by 0s. The elementary multiplets correspond to a single box filled with either a 1, $\bar{1}$ or 0. The problem occurs for the product

$$\boxed{\bar{1}} \cdot \boxed{1}.$$

This must be identified with the zero weight in the representation (2) and so must be

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline\end{array}.$$

Thus the stretched product for SO(n) Young tableaux, if it can be defined at all, must involve not only reordering, but also changes in the entries.

(II) There exist many other branching rules for which combinatorial procedures are known for computing the large-rank multiplicities which must be subsequently modified to recover the low-rank result King (1975). These may be used to obtain generating functions for these branching rules by generalising the procedures we have described. An example is provided by the branching rule $\mathrm{Sp}(2n) \times \mathrm{Sp}(2m) \subset \mathrm{Sp}(2n+2m)$ with three non-zero labels. In the notation of King (1975) the branching may be written $\langle\lambda\rangle \downarrow \Sigma_\sigma \langle\lambda / B\sigma\rangle \times \langle\sigma\rangle$. As a result we must combine the algorithms for the branchings $\mathrm{Sp}(2n) \subset \mathrm{SU}(2n)$ and $\mathrm{SU}(n) \times \mathrm{SU}(m) \subset \mathrm{SU}(n+m)$ to obtain the result. The corresponding tableaux thus contain letters corresponding to the first branching and numbers corresponding to the second. The resulting elementary tableaux may be found in table 11 and the syzygies in table 12. The resulting generating function is

$$AB\bar{B}C\overline{CD}E\bar{F}G(DIH + \bar{g}DI\bar{G} + fDHF + \overline{eE}IH + \overline{egE}I\bar{G} + \bar{e}f\bar{E}HF). \tag{6.1}$$

This generating function may also be obtained in a somewhat easier fashion by combining the two generating functions (4.3) and (4.4) for three non-zero labels, which are

$$(1-a_2)^{-1}(1-a_1b_1)^{-1}(1-a_3b_1)^{-1}(1-a_2b_2)^{-1}(1-a_3a_1b_2)^{-1}(1-a_3b_3)^{-1}$$

for $\mathrm{Sp}(2n) \subset \mathrm{SU}(2n)$ and

$$(1-c_1x_1)^{-1}(1-c_1y_1)^{-1}(1-c_2x_2)^{-1}(1-c_2y_2)^{-1}(1-c_2x_1y_1)^{-1}$$
$$(1-c_3x_3)^{-1}(1-c_3y_3)^{-1}(1-c_3x_2y_1)^{-1}(1-c_3x_1y_2)^{-1}$$

Table 11. Elementary multiplets for $\mathrm{Sp}(2n)\times\mathrm{Sp}(2m)\subset\mathrm{Sp}(2n+2m)$, three labels.

Multiplet	Auxiliary labels	Tableau	Multiplet	Auxiliary labels	Tableau
a	a_2	$\begin{smallmatrix}\boxed{1}\\ \boxed{2}\end{smallmatrix}$	$\bar{e}$	a_3y_1	$\begin{smallmatrix}\boxed{}\\ \boxed{1}\\ \boxed{2}\end{smallmatrix}$
b	a_3x_3	$\begin{smallmatrix}\boxed{a}\\ \boxed{b}\\ \boxed{c}\end{smallmatrix}$	f	a_2x_2	$\begin{smallmatrix}\boxed{a}\\ \boxed{b}\end{smallmatrix}$
$\bar{b}$	a_3y_3	$\begin{smallmatrix}\boxed{}\\ \boxed{}\\ \boxed{}\end{smallmatrix}$	$\bar{f}$	a_2y_2	$\begin{smallmatrix}\boxed{}\\ \boxed{}\end{smallmatrix}$
c	$a_3x_2y_1$	$\begin{smallmatrix}\boxed{}\\ \boxed{a}\\ \boxed{b}\end{smallmatrix}$	g	$a_3a_1x_2$	$\begin{smallmatrix}\boxed{a}\boxed{1}\\ \boxed{b}\phantom{\boxed{1}}\\ \boxed{2}\phantom{\boxed{1}}\end{smallmatrix}$
$\bar{c}$	$a_3x_1y_2$	$\begin{smallmatrix}\boxed{}\\ \boxed{}\\ \boxed{a}\end{smallmatrix}$	$\bar{g}$	$a_3a_1y_2$	$\begin{smallmatrix}\boxed{}\boxed{1}\\ \boxed{}\phantom{\boxed{1}}\\ \boxed{2}\phantom{\boxed{1}}\end{smallmatrix}$
d	a_1x_1	$\boxed{a}$	h	$a_2x_1y_1$	$\begin{smallmatrix}\boxed{}\\ \boxed{a}\end{smallmatrix}$
$\bar{d}$	a_1y_1	$\boxed{}$	i	$a_3a_1x_1y_1$	$\begin{smallmatrix}\boxed{}\boxed{1}\\ \boxed{a}\phantom{\boxed{1}}\\ \boxed{2}\phantom{\boxed{1}}\end{smallmatrix}$
e	a_3x_1	$\begin{smallmatrix}\boxed{a}\\ \boxed{1}\\ \boxed{2}\end{smallmatrix}$			

Table 12. Syzygies for $\mathrm{Sp}(n)\times\mathrm{Sp}(m)\subset\mathrm{Sp}(2n+2m)$.

1	$\bar{d}e = d\bar{e}$	3	$\bar{f}i = h\bar{g}$
2	$\bar{f}g = f\bar{g}$	4	$fi = hg$

for $\mathrm{SU}(n)\times\mathrm{SU}(m)\subset\mathrm{SU}(n+m)$. The required generating function may now be obtained by substituting $c_i = b_i^{-1}$ in the second generating function and then formally keeping the constant term with respect to the b_i in the product of the two generating functions. This procedure also yields the generating function (6.1).

(III) As a result of the remarkable similarity between the way multiplicities are calculated for branching $\mathrm{SU}(n)\times\mathrm{SU}(m)\subset\mathrm{SU}(n+m)$ and for Kronecker products in $\mathrm{SU}(n)$ the generating function (4.3) for branching multiplicities is also a generating function for Kronecker product multiplicities. Thus the presence of the term $a_4a_2b_3c_3$ indicates that the representation $(0, 1, 0, 1, \ldots)$ occurs in the product of $(0, 0, 1, 0, \ldots)$ with $(0, 0, 1, 0, \ldots)$ with multiplicity one.

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